Time and Space Optimal Exact Majority Population Protocols

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Abstract

In this paper we study population protocols governed by the *random scheduler*, which uniformly at random selects pairwise interactions between n agents. The main result of this paper is the first time and space optimal *exact majority population protocol* which also works with high probability. The new protocol operates in the optimal *parallel time* $O(\log n)$, which is equivalent to $O(n \log n)$ sequential *pairwise interactions*, where each agent utilises the optimal number of $O(\log n)$ states.

The time optimality of the new majority protocol is possible thanks to the novel concept of fixed-resolution phase clocks introduced and analysed in this paper. The new phase clock allows to count approximately constant parallel time in population protocols.

1 Introduction

The model of *population protocols* adopted in this paper was introduced in the seminal work of Angluin, Aspnes, Diamadi, Fischer, and Peralta [1, 2] for which the authors received 2020 Edsger W. Dijkstra Prize in Distributed Computing. Their model provides a universal theoretical framework for studying pairwise interactions within a large population of indistinguishable entities equipped with fairly limited storage, communication and computation capabilities. These entities are very often referred to as *agents*. The agents are often modelled as finite state machines. When two agents engage in a direct interaction their current states are modified according to the transition function that forms an integral part of the population protocol. In the *probabilistic variant* of population protocols, considered in [1] and adopted in this paper, the *random scheduler* selects for each interaction a pair of agents uniformly at random. In this variant, in addition to space utilisation referring to the maximum number of distinct *states* available to agents, one is also interested in the *running time* of the solution. In the original model [1] it is assumed that the number of states utilised by each agent is constant. In due course this restriction was lifted [10] allowing larger numbers of states (small functions of the population size) in the transition function. This model amendment enabled a wide range of studies on space/time trade-offs in probabilistic population protocols. More recent work in the field focuses on *parallel time* defined as the total number of pairwise interactions divided by the size (in our case n) of the population. The parallel time is also correlated with the local time of agents expressed in the number of individual interactions.

A population protocol *terminates with success* if the whole population eventually stabilises, i.e., it arrives at and stays indefinitely in the final configuration of states exhibiting the desired property of the solution. Among the most studied problems in population protocols are the leader election and the majority computation. In *leader election* in the final configuration a single agent is expected to conclude in the *leader* state and all other agents must stabilise in the *follower* states. The leader election problem received in recent years greater attention in the context of population protocols thanks to a number of important developments in closely related topics [17, 19]. In particular, the results from [17, 19] laid down the foundation for the proof that leader election cannot be solved in a sublinear time with agents utilising a fixed number of states [20]. In further work [7], Alistarh and Gelashvili studied the relevant upper bound, where they proposed a new leader election protocol stabilising in time $O(\log^3 n)$ assuming $O(\log^3 n)$ states per agent.

In more recent work [5] the authors consider more general trade-offs between the number of states utilised by agents and the time complexity of the solution. In particular, the authors provide a separation argument distinguishing between *slowly stabilising* population protocols which utilise $o(\log\log n)$ states and *rapidly stabilising* protocols with $O(\log n)$ states per agent. This result nicely coincides with another fundamental observation from [16] which states that population protocols utilizing $o(\log\log n)$ states are limited to semilinear predicates, while the availability of $O(\log n)$ states admits computation of symmetric predicates. More recent developments include also a protocol which elects the leader in time $O(\log^2 n)$ whp. and in expectation utilizing $O(\log^2 n)$ states [15]. This number of states was later reduced to $O(\log n)$ in [6] and [13] through the application of two types of synthetic coins. In [24, 25] the authors show that $O(\log^2 n)-$time leader election can be accomplished whp when each agent is utilising the asymptotically optimal number of $O(\log\log n)$ states. The running time of leader election was later improved to $O(\log n \log\log n)$ in [26] and very recently to the optimal time $O(\log n)$ in [14].

Another important direction in the ongoing studies on population protocols is the *majority problem* dating back to the seminal work [1], and coincidentally the main focus of this paper. In this problem in the final configuration all agents are expected to adopt one of the states representing the opinion of the majority. The first study on fast $O(\log n)-$time computation of *approximate majority* refers to the 3-state protocol [4]. In this problem one assumes a large enough *bias* of $\omega(\sqrt{n}\log n)$ expressing the difference between the numbers of agents sharing the opinion of the majority and the remaining agents. In contrast, in [21] and [30] one can find 4-state exact majority protocols which determine the opinion of the majority even if the original bias is 1. Unfortunately, the (parallel) time performance of these protocols grows rapidly when the bias gets smaller.

The most important developments in the *exact majority* computation include [6], where the authors show the lower bound $\Omega(\log n)$ on the number of states imposed on any protocol which stabilises in time $O(n^c)$, for any constant $c \leq 1$. They also match this bound from above by an algorithm which utilises $O(\log n)$ states at each agent, and stabilises in time $O(\log^2 n)$. The multi-stage majority protocols considered in [4] introduced the idea of alternating *cancellations* and *duplications*. This idea was later explored in [15] where the framework for agents migrating along $O(\log n)$ levels to increase the original bias was coined. Their $O(\log^2 n)$-time protocol was later improved to $O(\log^{5/3} n)$ in [12] and very recently to $O(\log^{3/2} n)$ in [11]. This left a very natural open problem whether a $O(\log n)$-time exact majority computation is feasible. Two excellent surveys [8, 23] provide a more detailed discussion on recent advances in the field of population protocols.

2 Preliminaries

In this section we provide further detail of the adopted model and the considered exact majority problem. We also discuss already known and novel tools which we use in our majority protocol.

2.1 The model

We consider population protocols defined on the complete graph of interactions where the *random scheduler* picks uniformly at random pairs of agents drawn from the population of size n. The agents are anonymous, i.e., they don't have identifiers. All agents draw their states from the same pool limited to $O(\log n)$ states. The protocol assumes that all agents start in the same initial state. Our protocol utilises the classical model of population protocols [1, 3] in which every interaction refers to an ordered pair of agents, namely ($\texttt{initiator}, \texttt{responder}$). Such ordering can be interpreted as follows. The random scheduler chooses first the initiator uniformly at random from the whole population and then the initiator chooses the responder also uniformly at random from the whole population. On the conclusion of each interaction the two participating agents change their original states a, b into a', b' according to a predefined *deterministic transition function* (forming a part of the actual algorithm), and such individual transitions will be denoted by $a, b \to a', b'$.

2.2 The exact majority problem

In the initial configuration agents can be *partial* by adopting opposite opinions $\mathcal{A}$ or $\mathcal{B}$, or they choose to be *objective*. In order to simplify the notation we refer to them as $\mathcal{A}$-agents, $\mathcal{B}$-agents (and together as *biased-agents*), and $\mathcal{O}$-agents respectively. The main task is to report which opinion of the two is more popular, i.e., whether there are more agents with opinion $\mathcal{A}$ or $\mathcal{B}$. And if there is a perfect balance between the two report *the tie*. The outcome of the majority computation is the final configuration in which all agents adopt the biased opinion of the majority, or all agents choose to be objective in the case of the tie. A crucial indicator in the majority problem is the absolute difference between the numbers of agents with opposite opinions called *the bias.* As discussed earlier there are known approximate majority protocols which assume a minimum size of the (opinion) bias, i.e., they report the correct answer when the bias is large enough. In this paper we consider the *exact majority* problem in which majority protocols are more sensitive, i.e., they give the correct answer for the bias as small as 1, in addition they distinguish the case of the tie.

2.3 The tools

In our solution we adopt the concept of cancelling and duplication introduced in [3] and successfully utilised in several recent exact majority algorithms [11, 12, 15, 28] based on a system of $\log n + 1$ virtual levels $L = \log n, \ldots, 0$. In this system all agents start with one of the two opinions $\mathcal{A}$ or $\mathcal{B}$ at level L and gradually migrate towards level 0. During this journey the agents can be stripped of their opinion during cancellation to become $\mathcal{O}$-agents. They can also adopt new opinions during duplication which in this paper is referred to as the *splitting process.* Please note that during the preprocessing a subpopulation of agents is chosen to run the phase clock and we refer to them as *clock-agents.* The lowest level available for the agents depends on the initial bias $\beta_0 > 0$, and the smaller this bias is the lower level is reached by the agents. In particular, one can observe that due to the doubling bias effect [15] the lowest level which all biased-agents can reach is $L - \log_2(\frac{n}{\beta_0}) \geq 0$. We say that the *mass* of a biased-agent currently residing at level i is 2^i, for any $i = \log n, \ldots, 0$, and the mass of every $\mathcal{O}$-agent is null. The *mass* of the population refers to the sum of the individual masses of all agents in the population. There are two leading transitions governing the distribution of the opinions, masses as well as the migration process itself:

- *Cancellation.* The mass of an agent is nullified during an interaction with an agent at the same level (of the same mass) but with the opposite opinion. On the conclusion of this transition both agents become $\mathcal{O}$-agents.

- *Splitting.* If a biased agent located at level l interacts with an $\mathcal{O}$-agent, they both adopt the opinion and a half of the mass of the biased agent, and in turn migrate to level $l - 1$.

We say that an event occurs *with high probability* (whp) if its probability is at least $1 - n^{-\eta}$ for all $n > n_\eta$. If the event refers to a behaviour of an algorithm, we say it occurs with high probability if the constants used in the algorithm can be fine-tuned so that the probability of this event is at least $1 - n^{-\eta}$. Analogously an event X occurs *with very high probability* (wvhp) if for any $b > 0$ there exists an integer n_b such that event X occurs with probability at least $1 - n^{-b}$ when $n > n_b$. In particular, if an event occurs with probability $1 - n^{-\omega(1)}$, it occurs with very high probability. In the analysis apart from standard Chernoff bounds we also utilise Chernoff-Janson inequality governing behaviour of geometric processes [27]. For the simplicity, in our calculations we often replace discrete summations by their integral counterparts when we replace equality sign with $\sim$ symbol, however, exactly the same results can be proved using more direct discrete calculations. We also use *the epidemic* proposed and analysed in [1] which guarantees rapid $O(\log n)$-time infection of (dissemination in) the whole population whp, and the fast 3-state approximate majority protocol [4], and a version of 4-state protocols from [21, 30] focused solely on cancellation.

There are known estimates for the epidemic processes by the logistic function, see, e.g., [31], however, without the required probabilistic guaranties. We show below three useful lemmas which provide good estimates on the time complexity of partial epidemic and splitting processes.

Lemma 1. *Let $0 < a < b < 1$ and $\varepsilon > 0$. Consider the epidemic protocol where initially subpopulation A, s.t., $|A| = a \cdot n$, contains all infected agents. The time t required to extend the subpopulation of infected agents from A to B, where $|B| = b \cdot n$, is $(1-\varepsilon)\hat{E} < t < (1+\varepsilon)\hat{E}$ wvhp, where*

$$\hat{E} = \frac{\ln(b) - \ln(1-b) - \ln(a) + \ln(1-a)}{2}.$$

Proof. Let $X_{a\cdot n} \ldots, X_{b\cdot n}$ be the sequence of gradually increasing infected subpopulations, where $X_{a\cdot n} = A$, $X_{b\cdot n} = B$, and $|X_i| = i$, for any $a \cdot n \le i \le b \cdot n$. Let T_i be the sequential time (the number of interactions) elapsed between formation of X_i and X_{i+1}. During this time the probability of having a new infection in a single interaction is $2(i/n)(1-i/n)$. Thus $ET_i = \frac{1}{2(i/n)(1-i/n)}$. And in turn the expected elapsed time between formation of A and subsequent formation of B is

$$ET = \sum_{i=an}^{bn-1} ET_i \sim n \int_a^b \frac{dx}{2x(1-x)} = \int_a^b \left(\frac{1}{2x} + \frac{1}{2(1-x)}\right) dx.$$

Thus

$$Et = \frac{ET}{n} \sim \left[\frac{1}{2}(\ln(x) - \ln(1-x))\right]_a^b = \hat{E}.$$

Finally, by Chernoff-Janson inequality [27] wvhp t is between $(1-\varepsilon)\hat{E}$ and $(1+\varepsilon)\hat{E}$. □

Corollary 1. *Given the precision constant $\varepsilon > 0$, a constant $0 < a < 1$ determining the initial size $a \cdot n$ of the infected subpopulation, and time $t > 0$ which permits the epidemic process to inflate the infected subpopulation to size $b \cdot n$. Then wvhp*

$$(1-\varepsilon)\frac{a}{1-a}e^{2t} < \frac{b}{1-b} < (1+\varepsilon)\frac{a}{1-a}e^{2t}.$$

Lemma 2. *Consider two disjoint subpopulations A and B of initial sizes $|A| = a \cdot n$, and $|B| = b \cdot n$, where $0 < b < a < 1$. An interaction between an agent in A with an agent in B is called* meaningful. *Such interaction results in elimination of both agents from their subpopulations. Thus after i meaningful interactions the numbers of agents in A and B are reduced to $a \cdot n - i$ and $b \cdot n - i$ respectively.*

The time t elapsed before $d \cdot n$ meaningful interactions occur satisfies wvhp $(1-\varepsilon)\hat{E} < t < (1+\varepsilon)\hat{E}$, where

$$\hat{E} = \frac{\ln(b) - \ln(a) - \ln(b-d) + \ln(a-d)}{2(a-b)}.$$

Proof. First we bound the expected time ET, where T refers to the number of interactions resulting in reduction of both subpopulations by $d \cdot n$. In addition, let T_i be the number of interactions between two consecutive meaningful interactions $i-1$ and i. Between T_{i-1} and T_i the probability of the next interaction being meaningful is $2(a-i/n)(b-i/n)$. So $ET_i = \frac{1}{2(a-i/n)(b-i/n)}$. Thus if ET is the expected number of all interactions in which $d \cdot n$ meaningful interactions occur we get

$$ET = \sum_{i=0}^{dn-1} ET_i \sim n \int_0^d \frac{dx}{2(a-x)(b-x)} = n \int_0^d \left(\frac{1}{2(a-b)(b-x)} - \frac{1}{2(a-b)(a-x)}\right) dx.$$

So

$$Et = \frac{ET}{n} \sim \left[\frac{1}{2(a-b)}(-\ln(b-x) + \ln(a-x))\right]_0^d = \frac{1}{2(a-b)}(\ln(b) - \ln(a) - \ln(b-d) + \ln(a-d))$$

By Chernoff-Janson bound, wvhp the time t deviates from $\hat{E}$ by at most $\varepsilon \cdot \hat{E}$, for any $\varepsilon > 0$. Which concludes the proof that

$$(1-\varepsilon)\hat{E} < t < (1+\varepsilon)\hat{E}.$$

holds wvhp. $\square$

Corollary 2. *Adopting notation from Lemma 2, the fraction d of n agents eliminated from each subpopulation A and B in a given time t satisfies wvhp*

$$(1-\varepsilon)\frac{b}{a}e^{-2(a-b)t} < \frac{b-d}{a-d} < (1+\varepsilon)\frac{b}{a}e^{-2(a-b)t}.$$

Lemma 3. *Let $0 < a, b_1, b_2, \varepsilon < 1$ be constants, where $b_1 > b_2$. Consider two subpopulations, A maintaining its size above $a \cdot n$, and B initially of size $b_1 \cdot n$. Any interaction between an agent in A with an agent in B is* meaningful, *and it forces the agent in B to leave its subpopulation.*

The parallel time t in which the size of subpopulation B is reduced from $b_1 n$ to $b_2 n$ wvhp satisfies $(1-\varepsilon)\hat{E} < t < (1+\varepsilon)\hat{E}$ where $\hat{E} = \frac{\ln b_1 - \ln b_2}{2a}$.

Proof. Let T_i, for all subsequent integer $i = b_1 \cdot n, \ldots, b_2 \cdot n$, be the number of all interactions between the $(b_1 \cdot n - i)$-th and $(b_1 \cdot n - i + 1)$-th meaningful interaction. Since any of the relevant T_i interactions is meaningful with probability at least $2a(i/n)$ we have $ET_i = \frac{1}{2a(i/n)}$. So the expected number of interactions to reduce the size of B from $b_1 n$ to $b_2 n$ is

$$ET = \sum_{i=b_2 n}^{b_1 n+1} ET_i = \sum_{i=b_2 n}^{b_1 n+1} \frac{1}{2a(i/n)} \sim n \int_{b_2}^{b_1} \frac{dx}{2ax} = \frac{n}{2a}[\ln x]_{b_2}^{b_1} = n\frac{\ln b_1 - \ln b_2}{2a}.$$

Since $Et \sim \hat{E}$, by Chernoff-Janson inequality the parallel time t elapsed during reduction of $|B|$ from $b_1 n$ to $b_2 n$ satisfies $(1-\varepsilon)\hat{E} < t < (1+\varepsilon)\hat{E}$. $\square$

3 Time and space optimal exact majority protocol

We encourage readers to familiarise themselves with the crude exact majority protocol discussed in the Appendix. This crude protocol performs well in experiments. It also helps to understand the basic principles behind our main time and space optimal exact majority protocol presented below.

The computation is done in two consecutive stages called the preprocessing discussed in Section 3.1 and the main majority protocol described later in Section 3.3.

3.1 The preprocessing

In this stage the whole population executes two protocols simultaneously: (i) the fixed-resolution clock discussed later in Section 3.2 utilizing $O(\log n)$ states and (ii) a version of 4-state majority protocol focused solely on opinion cancellation. If the original bias is of size at least $n/10$, after time $O(\log n)$ counted by the fixed-resolution clock the 4-state majority protocol computes the majority whp which concludes the exact majority computation process. Otherwise (the initial bias is smaller than $n/10$), after time $O(\log n)$ biased-agents are still present in the population, and this can be verified by the epidemic in time $O(\log n)$, also controlled by our new clock. In such case, all agents which originally had opinion $\mathcal{A}$ but lost it via cancellation join the clock subpopulation $\mathcal{C}$, where $|C|/n = c$. The remaining agents stay in the main population $\mathcal{M}$, where $|\mathcal{M}|/n = m$ and $m > c$. In the second and the main stage of the protocol either the majority (with the initial small bias) is determined or the tie case is reported, both whp.

3.2 Clock subpopulation $\mathcal{C}$ and fixed-resolution phase clock

In this section we introduce a novel concept of fixed-resolution phase clocks designed to count (approximately) a constant number of parallel time steps. In other words, the time elapsed between two consecutive ticks of the clock refers to a small constant. This is in a dramatic contrast to all currently known leader based [1, 2, 22] and leaderless [6, 11, 12, 18] phase clocks with resolutions $\Omega(\log n)$. In our majority protocol the clock is implemented in the clock subpopulation $\mathcal{C}$, where $0.45 \cdot n \leq |\mathcal{C}| < 0.5 \cdot n$. The content of $\mathcal{C}$ is determined during the preprocessing. And when $\mathcal{C}$ is formed the agents in $\mathcal{C}$ initiate the clock with value $L = O(\log n)$, and gradually reduce this value to 0. Since this clock has a fixed resolution the total parallel time of our protocol is also $O(\log n)$.

State space and transition function. At any time, each agent in $\mathcal{C}$ is located at some virtual level l, where $L = k \cdot \log n \geq l \geq 0$, where the integer constant k is later interpreted as the number of *minutes* in an *hour*. The state of each agent in $\mathcal{C}$ is represented by vector $(\mathcal{C}, \mathit{level})$, where the initial state is set to $(\mathcal{C}, L)$. Finally, since only the second attribute *level* requires $O(\log n)$ states, the total number of states utilised by the clock agents is $O(\log n)$.

There are only two transitions governing states of agents in subpopulation $\mathcal{C}$, including:

- **Dripping** $(\mathcal{C}, \mathtt{l}), (\mathcal{C}, \mathtt{l}) \xrightarrow{\text{prob}\,\mathtt{p}} (\mathcal{C}, \mathtt{l}-\mathtt{1}), (\mathcal{C}, \mathtt{l})$, where $p = 10^{-4}$, and
- **Epidemic** $(\mathcal{C}, \mathtt{l}), (\mathcal{C}, \mathtt{l}') \rightarrow (\mathcal{C}, \mathtt{min}(\mathtt{l}, \mathtt{l}')), (\mathcal{C}, \mathtt{min}(\mathtt{l}, \mathtt{l}'))$.

The first *dripping* transition enables rapid formation of a small fraction (of $|\mathcal{C}|$) at level $l-1$ as soon as almost all agents from $\mathcal{C}$ arrive at level l. At the same time this transition prevents from rapid migration to level $l-1$ when level l is smaller. The second *epidemic* transition allows to expand the small fraction at level $l-1$ to almost full $\mathcal{C}$ in a constant number of parallel steps.

3.2.1 Analysis of the constant resolution phase clock

We denote here by t the parallel time elapsed during some process and by T the corresponding number of interactions, i.e., $t = T/n$. Analogously we use Et to denote the expected elapsed parallel time and ET to denote the expected number of the relevant interactions, where also $Et = ET/n$.

We present now several lemmas which govern sizes of and changes at consecutive levels. Let $c_{l^-}(t)$ be the fraction of all clock-agents located in time t at levels l and below. Please note that this fraction can only increase as the agent migrate along decreasing levels. Recall also that $p = 10^{-4}$ refers to the the dripping probability observed at any level.

Lemma 4. *If $n^{-0.49} < c_{l^-}(t) < 0.2$, then $c_{(l-1)^-}(t) < 10pc_{l^-}(t)^2$ holds wvhp.*

Proof. The proof uses induction on (parallel) time t. Assume that the claim of the lemma holds at time $t-1$ and $c_{l^-}(t-1) = x'$. We show that the claim holds also at time t. Note that the monotonic growth of values $c_{(l-1)^-}(t)$ and $c_{l^-}(t)$ is affected by the epidemic and the dripping processes executed simultaneously. The minimum growth at level c_{l^-} between time $t-1$ and t is guaranteed by Corollary 1 which considers only the epidemic process. One can observe that

$$(1-\varepsilon)\frac{c_{l^-}(t-1)}{1-c_{l^-}(t-1)}e^2 < \frac{c_{l^-}(t)}{1-c_{l^-}(t)}$$

Thus we get $c_{l^-}(t) > c_{l^-}(t-1)e^2/0.8 > 5.5$. Using Corollary 1 one can also upperbound the growth inflicted by the epidemic progress on any subpopulation of size $a(t)$ as $a(t)/a(t-1) \le e^2(1+\varepsilon) < 7.5$. Thus the growth (caused by the epidemic) in relation to $c_{(l-1)^-}(t-1)$ is limited by the multiplicative factor of 7.5 wvhp. In addition, in one unit of time the dripping process contributes to the growth at most $1.1pc_{l^-}(t)^2$ wvhp, due to Chernoff bound. And this dripped (surplus) subpopulation can be augmented by the epidemic at most 7.5 times. In conclusion, using the inductive hypothesis and denoting $x = c_{l^-}(t)$ we get

$$c_{(l-1)^-}(t) < 7.5(c_{(l-1)^-}(t-1) + 1.1px^2) < 7.5(10px^2/5.5^2 + 1.1px^2) < 10px^2.$$

□

Lemma 5. *If $c_{l^-}(t) = 0.2$, then also $c_{l^-}(t+1.4) > 0.8$ wvhp.*

Proof. Assume we have $c_{l^-} = 0.2$ and the remaining fraction 0.8 of agents on levels above l. Consider the epidemic in which any interaction of an agent on any level above l with an agent on level l (or below) results in descending the agent from the higher level to level l (or below). No other interactions are considered. The expected time Et to grow c_{l^-} to 0.8 is by Lemma 1 bounded by

$$(1+\varepsilon)\frac{\ln(0.8) - \ln(1-0.8) - \ln(0.2) + \ln(1-0.2)}{2} > 1.4,$$

for any constant $\varepsilon > 0$. □

Lemma 6. *If $c_{l^-}(t) = 0.2$ and $c_{(l-1)^-}(t+\Delta t) = 0.2$, then also $\Delta t > 3.9$ wvhp.*

Proof. By Lemma 4 if $c_{l^-}(t) = 0.2$, then $c_{(l-1)^-}(t) < 0.00004$. Thus the time t the levels lower than l need to grow from fraction 0.00004 to 0.2 can be estimated as follows. The dripping from level l to $l-1$ occurs in each interaction with probability not greater than $0.0001(1-i/n)^2$, for as long as i agents reside below level l. In turn, for all subsequent integer $i \in (0.00004 \cdot n, 0.2 \cdot n)$, the probability p_i of moving the $i-$th agent from levels l or above to level $l-1$ is at most $2(1-i/n)(i/n)+(1-i/n)^2 \cdot 0.0001$, and the expected number of interactions to this happen is $ET_i \ge 1/p_i$. Finally, the expected time t to grow the fraction $c_{(l-1)^-} = 0.00004$ to $c_{(l-1)^-} = 0.2$ satisfies the following inequalities

$$\begin{aligned}
n \cdot t = ET_{0.00004n} + \cdots + ET_{0.2n} &\ge \sum_{i=0.00004n}^{0.2n} \frac{1}{2(1-i/n)(i/n) + (1-i/n)^2 \cdot 0.0001} \\
&\sim n \int_{0.00004}^{0.2} \frac{dx}{2(1-x)x + (1-x)^2/10000} = n \int_{0.00004}^{0.2} \frac{dx}{(1-x)(1.9999x + 0.0001)} \\
&= \frac{n}{2} \int_{0.00004}^{0.2} \left(\frac{1}{1-x} + \frac{1.9999}{1.9999x + 0.0001} \right) = \frac{n}{2} \left[-\ln(1-x) + \ln(1.9999x + 0.0001) \right]_{0.00004}^{0.2} \\
&= \frac{n}{2} (\ln(0.99996) - \ln(1-0.2) + \ln(1.9999 \cdot 0.2 + 0.0001) - \ln(1.9999 \cdot 0.00004 + 0.0001)) > 3.96n.
\end{aligned}$$

In conclusion, by Chernoff-Janson bound the parallel time t in which this expected progress occurs can be lowerbounded by 3.9 wvhp. □

Lemma 7. *If $c_{l^-}(t) = 0.2$ and $c_{(l-1)^-}(t+\Delta t) = 0.2$, then also $\Delta t < 6.5$ wvhp.*

Proof. By Lemma 5 we have $c_{l^-}(t+1.4) > 0.8$ whp. Because of the dripping process within time 0.5 we get $c_{(l-1)^-}(t+1.9) > 0.00003$. Now by Lemma 1 we can upperbound wvhp the time $\Delta t'$ in which $c_{(l-1)^-}$ grows from 0.00003 to 0.2, namely

$$\Delta t' < (1+\varepsilon)\frac{\ln 0.2 - \ln 0.00003 + \ln 0.99997 - \ln 0.8}{2} < 4.6.$$

Thus we conclude that $\Delta t < 1.4 + 0.5 + 4.6 = 6.5$ wvhp. $\square$

Finally, one can conclude this section with the following theorem.

Theorem 1. *The proposed clock protocol moves agents between consecutive levels in the clock in constant parallel time not shorter than 3.9 and not greater than 6.5 wvhp.*

We refer to the period from the theorem as one minute on the clock. Please note also that for a good fraction of each minute a vast majority of all clock-agents reside at the same level. Thus this protocol can be used as a constant resolution phase clock which is able to countdown from the initial time $O(\log n)$ to time 0. Note also that for the purpose of our majority algorithm we also distinguish hours on the clock where each hour is formed of a constant number $k(=5)$ of minutes.

3.3 The exact majority protocol overview

The 3-epoch protocol. The main part of the majority protocol operates in three consecutive epochs $\mathcal{E}_0, \mathcal{E}_1$ and $\mathcal{E}_2$, where $\mathcal{E}_0$ comprises levels $\log n+3, \ldots, \frac{2}{3}\log n+2$, $\mathcal{E}_1$ levels $\frac{2}{3}\log n+2, \ldots, \frac{1}{3}\log n+1$, and $\mathcal{E}_2$ the remaining levels $\frac{1}{3}\log n+1, \ldots, 0$. At the beginning of each epoch all agents in $\mathcal{M}$ start from the same (highest) level and gradually migrate towards the lowest. There are two main cases.

1. All biased-agents migrate to level 0 before the end of the epoch. This is never the case when the bias on the highest level is larger than $n^{2/3}$ (follows from the bias duplication [15]).

2. Some biased-agents remain above the lowest level at the end of the epoch. If the bias on the highest level is lower than a small constant fraction of $n^{2/3}$, this happens with negligible probability (follows from Theorem 3).

One can observe that if the initial bias is positive, then case (2) eventually happens, in epoch $\mathcal{E}_2$ at the latest. One can also distinguish between these two cases using one round of the epidemic (testing whether all biased-agents are at the lowest level) on the conclusion of an epoch. If the last (different to $\mathcal{E}_2$) epoch was of type (1) then the protocol proceeds to the next epoch. If the last epoch was of type (2) then agents restore their opinions from the beginning of the epoch and to conclude they run the slow 4-state majority protocol. (FIXME) In the remaining case, when epoch $\mathcal{E}_2$ is of type (1) our protocol reports the tie.

State space and transition function. At any time, each agent in $\mathcal{M}$ has either biased opinion $\mathcal{A}$ or $\mathcal{B}$ or is objective with opinion $\mathcal{O}$. At any time, an agent is located at some virtual level l, where $L = \log n + 3 \geq l \geq 0$. Thus the state of each agent in $\mathcal{M}$ can be described by vector $(\mathcal{M}, \texttt{opinion}, \texttt{level})$. As `level` requires $\log n + 4$ states, the overall number of states is $O(\log n)$.

In our solution we assume that only $\mathcal{O}$-agents communicate with clock-agents in $\mathcal{C}$. These agents migrate through $k \cdot \log n$ levels, for some positive integer constant k. The constant k refers to the number of minutes in an hour, i.e., the $\mathcal{O}$-agents in $\mathcal{M}$ are prompted to migrate to the lower levels only during full hours. The three transitions governing states of agents in subpopulation $\mathcal{M}$ include:

- **Clock updates**
  $(\mathcal{C}, l), (\mathcal{M}, \mathcal{O}, l') \rightarrow (\mathcal{C}, l), (\mathcal{M}, \mathcal{O}, \min\{\lfloor \frac{l}{k} \rfloor, l'\})$.

- **Cancelling**
  $(\mathcal{M}, \mathcal{A}, l), (\mathcal{M}, \mathcal{B}, l) \rightarrow (\mathcal{M}, \mathcal{O}, l), (\mathcal{M}, \mathcal{O}, l)$.

- **Splitting**
  $(\mathcal{M}, \mathcal{A}, l), (\mathcal{M}, \mathcal{O}, l') \rightarrow (\mathcal{M}, \mathcal{A}, l-1), (\mathcal{M}, \mathcal{A}, l-1)$, if $l > l'$.
  $(\mathcal{M}, \mathcal{B}, l), (\mathcal{M}, \mathcal{O}, l') \rightarrow (\mathcal{M}, \mathcal{B}, l-1), (\mathcal{M}, \mathcal{B}, l-1)$, if $l > l'$.

We also need extra transitions to manage the conclusion of each epoch. In this conference version of the we omit their formal definitions, relying on description of the interactions in the following sections.

3.4 Single epoch analysis

Recall that $\mathcal{M}$ refers to the main subpopulation and $\mathcal{C}$ to the clock subpopulation, both determined during the preprocessing, where $m = |\mathcal{M}|/n > c = |\mathcal{C}|/n$. Recall also that each epoch $\mathcal{E}_i$ is formed of $L+2$ levels populated by agents in $\mathcal{M}$: $L+1, L, \ldots, 1, 0$, where $L = \frac{1}{3}\log n$. We associate each of these levels with $k = 5$ levels (the number of minutes forming an hour) in the clock population $\mathcal{C}$. Moreover, at the end of each epoch we utilise some extra $O(\log n)$ clock levels not associated with any levels in $\mathcal{M}$. This is to distinguish via the epidemic between cases (1) and (2).

Recall that t denotes parallel time. By Lemmas 6 and 7 one minute on the clock governed by $\mathcal{C}$ takes parallel time between $3.9c^{-1}$ and $6.5c^{-1}$ whp. The epoch starts in time $t = 0$. We denote by $\bar{z}_{l^-}(t) = Z_l(t)/|\mathcal{M}|$, where $Z_l(t)$ is the number of transfers of $\mathcal{O}$-agents to or below level l from levels above l between time 0 and t. Similarly, by $\bar{w}_{l^-}(t)$ we denote the total number of biased-agents which migrated by splitting to levels l or below between time 0 and t divided by $|\mathcal{M}|$. Each splitting transition at level $l+1$ results in a transfer of two biased-agents to level l or below, and splitting at lower levels increases the number biased-agents by one. Both transfers are done at the expense of a single $\mathcal{O}$-agent at level l or below. This yields the following fact.

Fact 1. *We have $\bar{w}_{l^-}(t) \leq 2\bar{z}_{l^-}(t)$.*

The value $\bar{z}_{l^-}(t)$ can be upperbounded by the following lemma

Lemma 8. *If $n^{-0.49} < c_{l^-}(t) < 0.2$, then $\bar{z}_{l^-}(t) < 2.6 \cdot c_{l^-}(t)$ wvhp.*

Proof. Our proof uses induction on (parallel) time t. Assume, the thesis of the lemma holds at time $t - c^{-1}$ and $c_{l^-}(t - c^{-1}) = x'$. The minimum growth at level l and below in the clock subpopulation $\mathcal{C}$ between time $t - c^{-1}$ and t is guaranteed by Corollary 1 and can be bounded as in the proof of Lemma 4. Thus we get $c_{l^-}(t) > (1-\varepsilon)c_{l^-}(t - c^{-1})e^2/0.8 > 5.5x'$. The expected fraction of $\mathcal{O}$-agents in $\mathcal{M}$ migrating to level l or below between time $t - c^{-1}$ and t is at most $2c_{l^-}(t)$. So by Chernoff bound $\bar{z}_{l^-}(t) - \bar{z}_{l^-}(t - c^{-1}) < 2.1c_{l^-}(t)$ wvhp. In conclusion, using the inductive hypothesis we get

$$\bar{z}_{l^-}(t) < \bar{z}_{l^-}(t - c^{-1}) + 2.1c_{l^-}(t) \leq 2.6c_{l^-}(t - c^{-1}) + 2.1c_{l^-}(t) < 2.6c_{l^-}(t)/5.5 + 2.1c_{l^-}(t) < 2.6c_{l^-}(t).$$

□

Let $\mathcal{M}_{l^+}$ be the set of all biased-agents located at levels l and higher. Let $l(a)$ be the level of agent $a \in \mathcal{M}_{l^+}$ and $\mu_l[a] = 2^{l(a)-l}$ be the (relative) *mass* of a wrt to level $l \le l(a)$. By μ_{l^+} we denote the total (relative) mass of agents in $\mathcal{M}_{l^+}$ wrt level l which is defined as $\mu_{l^+} = \sum_{a \in \mathcal{M}_{l^+}} \mu_l[a]$. Denote also the mass μ_{l^+} at time t by $\mu_{l^+}(t)$. The total mass of agents at time t is defined as $\mu(t) = \mu_{0^+}(t)$. Note that the total mass $\mu(t)$ can only get smaller during execution of the protocol.

Let β denote the bias in the population at the start of the considered epoch. This bias is preserved during the epoch in the form of this invariant.

$$2^L\beta = \left| \sum_{\mathcal{B}-agent\ b \in \mathcal{M}_l} \mu[a] - \sum_{\mathcal{A}-agent\ a \in \mathcal{M}_l} \mu[a] \right|$$

In order to measure the progress of migration observed in $\mathcal{M}_{l^+}$ we define the *potential* of this subpopulation wrt level l. Let $\phi_l[a] = 4^{l(a)-l}$ denote the *potential* of agent $a \in \mathcal{M}_{l^+}$ wrt to level l. The potential of $\mathcal{M}_{l^+}$ is defined as $\phi_{l^+} = \sum_{a \in \mathcal{M}_{l^+}} \phi_l[a]$. Both values μ_{l^+} and ϕ_{l^+} evolve in time t which is denoted $\mu_{l^+}(t)$ and $\phi_{l^+}(t)$ respectively. One can observe $\mu_{l^+}(t) \le \phi_{l^+}(t)$.

In due course, we will prove that if $\beta \cdot 2^L < 0.01 \cdot |\mathcal{M}|$ then on the conclusion of the epoch all biased-agents arrive at level 0 resulting in the final bias $2^L\beta$. We start with the proof of Theorem 2 which states that for all $l \in \{L, \dots, 0\}$ certain conditions observed at some point at level l are replicated at level $l-1$ within a fixed period of time called a *cycle* C_l, and each cycle C_l takes five minutes (on the phase clock).

Theorem 2. *For all $l = L, \dots, 0$ the following two conditions are replicated within C_l, wvhp:*

1. $\phi_{(l+1)^+}(t) < \frac{1}{1000}|\mathcal{M}|$
2. $\mu(t) < 0.1 \cdot 2^{L-l} \cdot |\mathcal{M}|$

Proof. The proof is done by induction on l going from L down to 0. Note that the two conditions in the Theorem are true at the beginning of the epoch and in turn C_L. The potential of levels above L is 0 and the condition $\mu_{l^+}(0) \le 0.1 \cdot 2^L \cdot |M|$ is inherited either from the preprocessing or from the previous epoch. We denote by t_0 the time in which the first minute of the cycle begins. It is the same time when the hour level l (l is multiple of k) of the clock satisfies $c_{l^-}(t_0) = 0.2$. By Lemma 5 this level satisfies also $c_{l^-}(t_1) > 0.8$ when time $t_1 = t_0 + 1.4c^{-1}$. We show now that the first four minutes of the cycle is enough to secure replication of the two conditions. And this time is at least $4 \cdot 3.9c^{-1} - 1.4c^{-1} = 14.2c^{-1}$.

In order to prove the upper bound we need the following support lemmas. In Lemma 9 we show that the time allowing to secure at level l a stable fraction of $0.75 \cdot |\mathcal{M}|$ consisting of $\mathcal{O}$-agents is $2c^{-1}$. In Lemma 10 we prove that the time to reduce the potential adequately is $\frac{6.6n}{|\mathcal{M}|} \le 6.6c^{-1}$. Finally in Lemma 11 we show that the time needed to reduce the value of $\mu(t)$ to $0.1|\mathcal{M}|2^{L-l}$ is $\frac{5.5n}{|\mathcal{M}|} \le 5.5c^{-1}$. This gives the total time

$$2c^{-1} + 6.6c^{-1} + 5.5c^{-1} < 14.2c^{-1}.$$

$\square$

Lemma 9. *At least $0.75 \cdot |\mathcal{M}|$ $\mathcal{O}$-agents are observed at level l between times $t_2 = t_1 + 2c^{-1}$ and t_4, where t_4 is the end of the 4-th minute (on the phase clock).*

Proof. The number of biased-agents in $\mathcal{M}$ located below level l in time t_1 is at most $\bar{w}_{(l-1)^-}(t_1)|\mathcal{M}| < 0.0001 \cdot |\mathcal{M}|$. The mass of $\mathcal{M}_{l^+}(t_1)$ is at most $0.1 \cdot 2^{L-l+1}|\mathcal{M}|$ by the inductive hypothesis, see Theorem 2. Thus after time t_1 set $\mathcal{M}_{l^+}(t_1)$ contains at most $0.2 \cdot |\mathcal{M}|$ agents. Assume that exactly $a \cdot |\mathcal{M}|$ biased-agents are in $\mathcal{M}_{l^+}(t_1)$. Thus in time t_1 the number of all $\mathcal{O}$-agents is at least $(1 - a - 0.0001)|\mathcal{M}|$, and these agents form set $\mathcal{Z}_0$.

Beyond time t_1 the agents from $\mathcal{Z}_0$ can reduce their levels by interacting with clock subpopulation. They also contribute to splitting. Using Lemma 3, we bound the time period in which at most $0.049|\mathcal{M}|$ agents from $\mathcal{Z}_0$ have no interaction with the clock-agents located on levels l and below, by

$$(1+\varepsilon)\frac{\ln(1-a-0.0001)m - \ln 0.049m}{2 \cdot 0.8c} < (1+\varepsilon)\frac{\ln m - \ln 0.049m}{2 \cdot 0.8c} < 2c^{-1}.$$

All agents from $\mathcal{Z}_0$ which take part in interactions with clock level l or lower before time $t_2 = t_1 + 2c^{-1}$ form set $\mathcal{Z}_1$, where $|\mathcal{Z}_1| > (0.9509 - a)|\mathcal{M}|$. Note that between t_1 and t_4 at most $\bar{z}_{(l-1)^-}(t_4)|\mathcal{M}| < 0.0001 \cdot |\mathcal{M}|$ agents in $\mathcal{Z}_1$ migrate below level l. Thus the remaining agents must either stay on level l or contribute to splitting. There can be at most $(0.2 - a)|\mathcal{M}|$ splits with $\mathcal{O}$-agents on or above level l beyond time t_1, which is guaranteed by the mass of $\mathcal{M}_{l^+}(t_1)$ smaller or equal to $0.1 \cdot 2^{L-l+1}|\mathcal{M}|$, and also $|\mathcal{M}_{l^+}(t_1)| = a|\mathcal{M}|$. Thus between time t_2 and t_4 there are at least

$$|\mathcal{Z}_1| - (0.2 - a)|\mathcal{M}| - \bar{z}_{(l-1)^-}(t_4)|\mathcal{M}| > 0.75|\mathcal{M}|$$

$\mathcal{O}$-agents in $\mathcal{Z}_1$ staying at level l between times t_2 and t_4. □

Let $t_3 = t_2 + \frac{6.6n}{|\mathcal{M}|}$ and $\vec{\tau} = (t_2, t_3)$.

Lemma 10. *At time t_3 the potential $\phi_{(l+1)^+}(t_3) < |\mathcal{M}|/1000$ wvhp.*

Proof. From time t_2 a fraction of at least 0.75 of agents in $\mathcal{M}$ are $\mathcal{O}$-agents at level l, see Lemma 9. These $\mathcal{O}$-agents interact with biased-agents in $\mathcal{M}_{(l+1)^+}(t)$ causing them to split. In time t_2 the number of biased-agents at level $l+1$ is upperbounded by $0.1|\mathcal{M}|$ by the inductive hypothesis wrt the mass $\mu(t_0)$, see Theorem 2. So using the inductive hypothesis wrt $\phi_{(l+2)^+}(t_2)$ we get $\phi_{(l+1)^+}(t_2) < 0.1|\mathcal{M}| + 4\phi_{(l+2)^+}(t_2) \le 0.104|\mathcal{M}|$. During each interaction of period $\vec{\tau}$ any biased-agent does not interact with a $\mathcal{O}$-agent on level l with probability at most $1 - \frac{3|\mathcal{M}|}{2n^2}$. Thus in any subperiod of $\vec{\tau}$ of length $\frac{n}{20|\mathcal{M}|}$ the probability that a biased-agent does not have such an interaction is at most $(1 - \frac{3|\mathcal{M}|}{2n^2})^{\frac{n \cdot n}{20|\mathcal{M}|}} \sim e^{3/40}$, as one unit of time refers to n interactions.

The potential of a biased-agent gets reduced during any interaction with an $\mathcal{O}$-agent at level l. In a subperiod of length $\frac{n}{20|\mathcal{M}|}$ a biased-agent contributes to such interaction with probability $\sim 1 - e^{-3/40}$. Thus the average reduction of the potential during this subperiod is by at least a fraction of $\sim (1 - e^{-3/40})/2$ of the initial potential. This is because any splitting transition reduces the potential of the biased-agent by a factor of 2, and this reduced potential is shared with the $\mathcal{O}$-agent evenly. In other words the overall potential drops on average at least $\sim 1 - (1 - e^{-3/40})/2 = (1 + e^{-3/40})/2$ times.

Each agent in the epoch has its potential in $[1, 4^L] \subseteq [1, 4n^{2/3}]$. By dividing individual potentials by $4n^{2/3}$ we get random variables in $[0, 1]$ which allows utilisation of Chernoff bound. By Chernoff inequality, since $0.965 > (1 + e^{-3/40})/2$ we get for any time $t \in \vec{\tau}$ reduction of the overall potential wvhp bounded by

$$\phi_{(l+1)^+}\left(t + \frac{n}{20|\mathcal{M}|}\right) \le 0.965\phi_{(l+1)^+}(t).$$

We also note that $0.965^{-132} > 104$, thus the time $\frac{6.6n}{|\mathcal{M}|}$ is sufficient to reduce the potential 104 times to $\phi_{(l+1)^+}(t_3) < |\mathcal{M}|/1000$. $\square$

Assume that $2^L\beta < 0.01|\mathcal{M}|$. Using this assumption we prove that during period (t_3, t_4) the overall mass of all biased-agents gets reduced to at most $0.1|\mathcal{M}|2^l$.

Lemma 11. *Beyond t_3, the time needed to reduce the overall mass (of biased-agents) from $0.1|\mathcal{M}|2^{l+1}$ to $0.1|\mathcal{M}|2^l$ is at most $5.5\frac{n}{|\mathcal{M}|}$ wvhp.*

Proof. In the worst case $\mu(t_3) = 0.1|\mathcal{M}|2^{l+1}$, all biased-agents outside of level l have the opinion of the minority, and the the difference between masses of $\mathcal{B}$-agents and $\mathcal{A}$-agents has the maximum allowed value $2^L\beta = 0.01|\mathcal{M}|$. And this happens at level l there is at most the fraction of

$$\frac{\mu(t_3)}{2 \cdot 2^l|\mathcal{M}|} + \frac{2^{L-l}\beta}{2} \leq 0.1 + 0.005 = 0.105$$

of biased-agents in the majority. This difference between the numbers of $\mathcal{B}$-agents and $\mathcal{A}$-agents at level l expressed as a fraction of $|\mathcal{M}|$ has to be the largest possible to maximize the time. This difference can be upperbounded with the help of the inequality $\mu_{(l+1)^+}(t_3) \leq \phi_{(l+1)^+}(t_3) \leq 0.001|\mathcal{M}|$. We get

$$2^L\beta + \frac{2\mu_{(l+1)^+}(t_3)}{|\mathcal{M}|} + \frac{\bar{w}_{(l-1)^-}(t_4)}{2} \leq 0.01 + 0.002 + 0.00004 \cdot 2.6 \leq 0.013.$$

We now give the upper bound on the time needed to reduce the number of biased-agents at level l with these worst case parameters by fraction 0.05. Using Lemma 2 we conclude that this time is smaller than

$$(1+\varepsilon)\frac{\ln(0.092\frac{|\mathcal{M}|}{n}) - \ln(0.105\frac{|\mathcal{M}|}{n}) - \ln(0.042\frac{|\mathcal{M}|}{n}) + \ln(0.055\frac{|\mathcal{M}|}{n})}{2 \cdot 0.013\frac{|\mathcal{M}|}{n}} < 5.5\frac{n}{|\mathcal{M}|}.$$

$\square$

Theorem 3. *If $2^L\beta < 0.01|\mathcal{M}|$, then whp the epoch ends with all biased-agents on level 0.*

Proof. By Theorem 2 within the first $5(L+1)$ minutes we reach the condition $\phi_{1^+}(t_2) < |\mathcal{M}|/1000$. By Lemma 9 we also get a subpopulation of at least $0.75|\mathcal{M}|$ $\mathcal{O}$-agents at level 0 by the end of the epoch. We now estimate the expected value $E\phi_{l^+}(t_2 + \Delta t)$ for any time period Δt.

During an interaction the probability that a biased-agent interacts with a $\mathcal{O}$-agent at level 0 is $0.75\frac{|\mathcal{M}|}{n}$. If this is biased-agent is at level 1 or above, its potential is reduced by at least half. Adopting now $t + 1/n$ as time t plus one interaction we get

$$E\phi_{1^+}(t + 1/n) \leq E\phi_{1^+}(t)\left(1 - 0.75\frac{|\mathcal{M}|}{n^2}\right).$$

This implies

$$E\phi_{1^+}(t + \Delta t) \leq \left(1 - 0.75\frac{|\mathcal{M}|}{n^2}\right)^{n\Delta t}\phi_{1^+}(t) \sim e^{-0.75\frac{|\mathcal{M}|}{n}\Delta t}\phi_{1^+}(t).$$

Thus if we adopt $\Delta t = \frac{\eta \log n}{0.75\frac{|\mathcal{M}|}{n}} + \frac{\log n}{0.75\frac{|\mathcal{M}|}{n}}$, then $\phi(t_2 + \Delta t) \leq 0.001|\mathcal{M}|e^{-\eta \log n - \log n} < n^{-\eta}$. And since the expected potential is less than $n^{-\eta}$, whp there are no biased-agents above level 0 after the extra time Δt beyond $4L$ minutes required to make complete L cycles. And this extra time Δt can be implemented by utilizing additional $\lceil \Delta t/(2 \cdot 3.9) \rceil$ minutes on the clock. $\square$

In contrast to the previous theorem, which assures wvhp that all biased-agents get to level 0 when the $2^L\beta$ is a small constant fraction of $|\mathcal{M}|$ we have the following simple fact for $2^L\beta$ being larger than $|\mathcal{M}|$. This fact follows from the impossibility of distributing a large bias on level 0 where all biased-agents have mass 1.

Fact 2. *If $2^L\beta > |\mathcal{M}|$, then on the conclusion of the epoch there are some biased-agents left on levels higher than* 0*.*

3.5 The 3-epoch protocol

In this section we conclude with the main theorem stating that the 3-epoch protocol described in section 3.3 determines the majority or declares the tie in time $O(\log n)$ with whp.

Theorem 4. *The 3-epoch protocol solves the exact majority problem in time $O(\log n)$ whp.*

Proof. The proof follows from the logical structure of the 3-epoch protocol, Theorem 3 and Fact 2. □

4 Conclusion

In this paper we proposed and analysed (Theorem 4) the first space and time optimal exact majority population protocol which concludes the search for such protocol initiated in [1]. One of the major contributions of this paper is the fixed-resolution phase clock able to count (approximately) a constant number of parallel rounds (Theorem 1). Several problems remain open, including the formal analysis of the crude algorithm discussed in the Appendix. Another interesting problem is the question whether one can implement our fixed-resolution phase clock utilising $o(\log n)$ space without loosing quality guaranties.

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5 Appendix

5.1 Crude Exact Majority Algorithm

All agents start at the highest (virtual) level $\log n$. The state of each agent is formed of the pair ($\texttt{opinion}, \texttt{level}$), which components correspond to the current opinion and the current level of the agent respectively. Thus in the starting configuration each agent is in one of the three states:

- $(\mathcal{B}, \log n)$,
- $(\mathcal{A}, \log n)$, or
- $(\mathcal{O}, \log n)$.

The agents execute the majority protocol via consecutive pairwise interactions where each pair of agents is chosen uniformly at random. On the conclusion of each interaction the relevant pair of agents update their states utilising a symmetric transition function defined in the section below.

5.1.1 Transition function

We consider four types of *meaningful interactions* during which the states of agents change.

Bias-cancelling

$(\mathcal{B}, i), (\mathcal{A}, i) \to (\mathcal{O}, i), (\mathcal{O}, i)$,
$(\mathcal{A}, i), (\mathcal{B}, i) \to (\mathcal{O}, i), (\mathcal{O}, i)$.

Bias-splitting

$(\mathcal{B}, i), (\mathcal{O}, j) \to (\mathcal{B}, i-1), (\mathcal{B}, i-1)$, for any $j < i$,
$(\mathcal{A}, i), (\mathcal{O}, j) \to (\mathcal{A}, i-1), (\mathcal{A}, i-1)$, for any $j < i$.

$\mathcal{O}$-dripping

$(\mathcal{O}, i), (\mathcal{O}, i) \to (\mathcal{O}, i+1), (\mathcal{O}, i)$ with a small constant probability $p \approx 10^{-4}$.

$\mathcal{O}$-epidemic

$(\mathcal{O}, i), (\mathcal{O}, j) \to (\mathcal{O}, \mathtt{min}(i, j)), (\mathcal{O}, \mathtt{min}(i, j))$.

5.1.2 Intuition

Recall that the main idea utilised in recent exact majority protocols is to migrate all agents trough $\Omega(\log n)$ levels to inflate the initial bias. In the past work migration between two consecutive levels required $\omega(1)$ parallel time which results in the overall $\omega(\log n)$ parallel time (with the currently best known time complexity $O(\log^{3/2} n)$ [11]) for the exact majority computation. In this section we provide some experimental evidence indicating that the crude majority protocol migrates the whole population of agents between any two (with the exception of the lowest levels where the bias is close to n) consecutive levels in $O(1)$ parallel time. To be exact, the population is always spread along several consecutive levels but at *crucial periods* almost all agents belong to the same level.

In the crude protocol the whole population is "pulled-down" the levels by neutral $\mathcal{O}$-agents which are on the forefront of the migration wave, where by the forefront we understand the level with the currently largest number of $\mathcal{O}$-agents. This downward movement of $\mathcal{O}$-population is propelled by $\mathcal{O}$-dripping (to the next level) and further enhanced by $\mathcal{O}$-epidemic to bring $\mathcal{O}$-agents to the forefront as quickly as possible. Thanks to the (downward) progress dwindling effect of $\mathcal{O}$-dripping the growth of the lowest levels (below the current forefront) is fairly limited. This way we avoid the effect of detrimental spread of agents across too many ($\Omega(1)$) levels of comparable size. However, when a new forefront is formed at level i, i.e., level i contains close to n $\mathcal{O}$-agents, due to $\mathcal{O}$-dripping effect level $i-1$ is already of linear size in n. And in turn thanks to $\mathcal{O}$-epidemic the system needs only $O(1)$ parallel time to migrate the forefront to level $i-1$.. The remaining non $\mathcal{O}$-agents agents stay just above the forefront thanks to the bias-splitting effect. In addition, a good balance between $\mathcal{O}$-agents and the rest of the population is maintained by bias-cancelling. I.e., when the number of $\mathcal{O}$-agents drops too low color-cancelling starts dominating color-splitting and vice versa.

We support our claim by several experiments, and we show in the main part of our paper that a fine tuned version of this protocol gives provable $O(\log n)$ parallel time guarantees with high probability.

5.1.3 Experiments

We experimented with populations of size up to 10 million and we used the standard Java pseudo random number generator to select pairs of agents for the consecutive interactions.

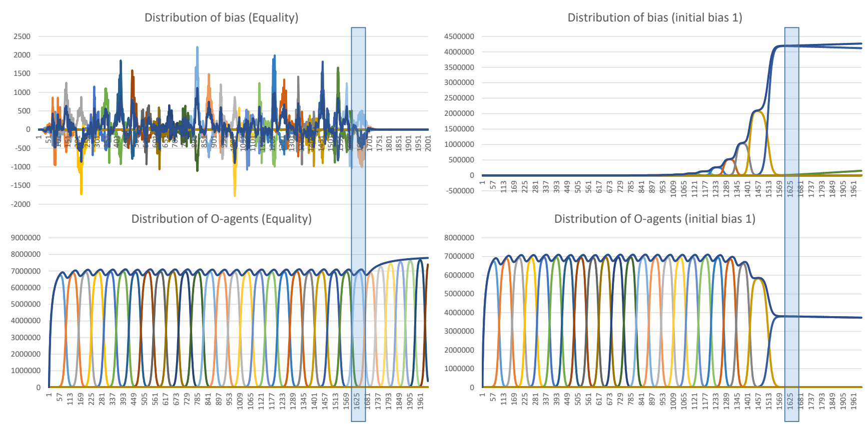

Figure 1: Distributions of the bias and $\mathcal{O}$-agents across the consecutive levels for the cases with equality (on the left) and a small initial bias (on the right).

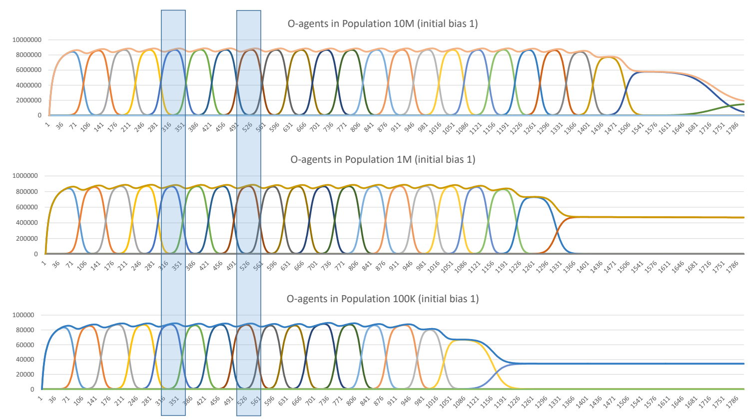

Figure 2: Distributions of $\mathcal{O}$-agents across the consecutive levels for different sizes of the population.

In our experiments we allow agents to migrate below level 0 setting the absolute limit at level $-\frac{\log n}{4}$ for agents carrying bias. This level can be reached by biased-agents only in the case of the tie. Otherwise, the bias will eventually suppress (fast) formation of the new forefronts, and this will happen at the latest at level 0. Note that in each picture the color of a curve refers to one specific level. On the X-axis one can find the parallel time expressed in minutes, where each minutes corresponds to $n/10$ interactions, i.e., $1/10$ of the parallel time unit. The Y-axis refers to the number of agents where in Figure 1 the population is of size 8M, and Figure 2 the population sizes vary between 10M, 1M and 100K.

We first consider behaviour of the population in the case of the tie, see the two pictures on the left in Figure 1. One can observe that the sub-population of $\mathcal{O}$-agents migrates swiftly along the consecutive levels amounting to about 90% of the whole population of 8M. At the same time the local bias (caused by some delayed biased-agents transfers between the consecutive levels) is bounded by at most a couple of thousands (order of the square root of the size of the population). The situation changes when all biased-agents arrive at level $-\frac{\log n}{4}$ where the bias drops to 0 and the final cancellation between biased agents results in increasing (and tending to 100%) number of $\mathcal{O}$-agents. These $\mathcal{O}$-agents continue to migrate and in fact are a prototype (in which formation of the forefront at level i refers to time i) of the fixed-resolution phase clock adopted in our main exact majority protocol. If we stop migration when all $\mathcal{O}$-agents arrive at level $-2logn$ the number of $\mathcal{O}$agents in the whole system is $n - o(n)$ with only few non $\mathcal{O}$-agents remaining. To conclude the exact majority computation We can run any known constant-state majority protocol between $\mathcal{O}$-, $\mathcal{A}$- and $\mathcal{B}$-agents to compute $\mathcal{O}$-majority indicating the tie case.

The situation is very different when we start with a (small) bias. In such case one can observe that eventually at some level $i \geq 0$ the bias amounts to more than 50% of the population. In such case, the migration process slows down dramatically as the rate with which $\mathcal{O}$-agents migrate (due to $\mathcal{O}$-dripping an the epidemic) to level $i - 1$ is the same as the bias (now in majority) at level l consumes new $\mathcal{O}$-agents at $l - 1$ during the splitting process. And when this happens, the system experiences a slow transfer of the bias between levels i and $i - 1$, the new forefront of $\mathcal{O}$-agents is never formed and eventually all $\mathcal{O}$-agents are replaced by the bias.

The gradual loss of $\mathcal{O}$-agents which propel timely migrations resembles the slowing down effect in ancient water clocks, due to constantly decreasing amount of water in their tanks [29]. To counterpart this undesired effect in our main exact majority protocol we split the agents into two subpopulations of comparable size, $\mathcal{C}$ responsible for the clock actions, i.e., timely counting and the forefronts formation, and $\mathcal{M}$ responsible for migration of biased-agents and in turn for the increasing in size bias.

In Figure 2 we provide some visual evidence that the rate (ticks of the clock) with which the consecutive forefronts based on $\mathcal{O}$-agents are formed is independent on the size of the population.